KINEMATIC INTERPRETATION OF THE MOTION OF A BODY WITH A FIXED POINT

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The analytical expressions obtained for particular solutions of the investigated problem are usually very complicated. Hence the importance of the kinematic interpretation of the solutions obtained [1].

To realize the well-known representation of rolling without slipping of a cone on another stationary cone it is necessary to know the directrix of the stationary cone. In a few particular cases the equation of this line is known. Poinsot [2] obtained its equation for the Euler solution, and Darboux [3] for the Lagrange solution.

In this paper the required equation is found for the general case of a body with a fixed point. As an example we have selected the solution of Chaplygin [2] because it is easy to demonstrate with it all the advantages of knowing the equation of the directrix, and also because the remaining relationships for this solution have been found by Chaplygin.

1. The moving hodograph of the angular velocity is described by Equations

$$\omega_i = \omega_i(\sigma) \qquad (i = 1, 2, 3) \tag{1.1}$$

giving the components of the angular velocity in the coordinate system moving with the body and depending on the variable σ. For example, in the case of a body under action of gravitational forces we can find the relations $y = y(x)$ and $z = z(x)$ from Equations (1.13) in [5], and then Formulas

$$\omega_1(x) = ax + b_1 y(x) + b_2 z(x)$$

$$\omega_2(x) = a_1 y(x) + b_1 x, \qquad \omega_3(x) = a_2 z(x) + b_2 x$$

given in the same reference, determine the moving hodograph. The relation between x and t is obtained by quadrature from (1.9) as shown in [5].

Knowing (1.1) we can show that there is a vector depending on σ which preserves a fixed direction with respect to a stationary reference system. In the case of a heavy rigid body this is the vector γ shown in [5] as a function of x . In the problem of Zhukovskii [6] on the inertial motion of a body having its fixed point in a liquid filled cavity this is the angular momentum vector of the system. In the problem of the motion of a body in the Newtonian force field [7] this is the vector along the line joining the fixed point with the center of attraction.

Let the unit vector along this fixed direction be ν and its components along the axes moving with the body be $\nu_i = \nu_i(\sigma)$. We make the axis ζ of the fixed cylindrical coordinate system, to coincide with this unit vector. The axial and the radial components of the angular velocity are determined from Formulas

$$\omega_\zeta(\sigma) = \boldsymbol{\omega}(\sigma)\cdot\boldsymbol{\nu}(\sigma), \qquad \omega_\rho(\sigma) = |\boldsymbol{\nu}(\sigma)\times\boldsymbol{\omega}(\sigma)| \tag{1.2}$$

The rate of change of the unit vector $\boldsymbol{\nu}$ with espect to the body is given by Equation

$$d\boldsymbol{\nu}/dt = \boldsymbol{\nu}\times\boldsymbol{\omega} \tag{1.3}$$

hence

$$\omega_\rho = |d\boldsymbol{\nu}/dt| \tag{1.4}$$

To determine the stationary hodograph of the angular velocity it is necessary as a supplement to (1.2) to show the dependence on σ of the third cylindrical coordinate, which is the angle α . We shall derive this dependence.

The tip of the vector $\boldsymbol{\omega}$ moves both on the moving and on the stationary hodograph with the velocity $d\boldsymbol{\omega}/dt$ whose projection on the normal to the plane containing the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\nu}$ equals

$$\frac{\boldsymbol{\nu}\times\boldsymbol{\omega}}{|\boldsymbol{\nu}\times\boldsymbol{\omega}|}\frac{d\boldsymbol{\omega}}{dt}$$

In the cylindrical coordinates the circular velocity component is given by Formula $\omega_\rho\, d\alpha/dt$, hence

$$\omega_\rho\frac{d\alpha}{dt} = \frac{\boldsymbol{\nu}\times\boldsymbol{\omega}}{|\boldsymbol{\nu}\times\boldsymbol{\omega}|}\frac{d\boldsymbol{\omega}}{dt} \tag{1.5}$$

or, in terms of the components and by (1.2)

$$\omega_\rho^2\frac{d\alpha}{d\sigma} = \begin{vmatrix} \nu_1(\sigma) & \nu_2(\sigma) & \nu_3(\sigma) \\ \omega_1(\sigma) & \omega_2(\sigma) & \omega_3(\sigma) \\ d\omega_1/d\sigma & d\omega_2/d\sigma & d\omega_3/d\sigma \end{vmatrix} \tag{1.6}$$

By using (1.5), (1.3) and (1.4) we can introduce a compact notation for Equation (1.6)

$$\frac{d\alpha}{dt} = \frac{d\boldsymbol{\nu}\cdot d\boldsymbol{\omega}}{d\boldsymbol{\nu}\cdot d\boldsymbol{\nu}}$$

By (1.2) and (1.6) the stationary hodograph of the angular velocity is fully determined. On this curve rolls without slipping the curve which moves the body and is given by (1.1). We have in (1.2) a parametric equation of a curve in the $\omega_\rho\omega_\zeta$ plane. This curve, in the fixed space $\omega_\xi\omega_\eta\omega_\zeta$, can be interpreted as a meridian of a surface of revolution. In cases where this is sufficiently straight, the motion of the body can be represented as rolling without slipping of the curve connected with the body (1.1) on the fixed surface of revolution. To determine the line traced by the moving hodograph on this surface we must use Equation (1.6).

In Euler's solution this surface of revolution is a plane and (1.6) is the equation of the herpolhode. In Lagrange's solution this surface of revolution is a sphere and (1.6) is the equation of Darboux [3].

2. Equations

$$A\frac{dp}{dt} = (B-C)\,qr + e_2\gamma_3 - e_3\gamma_2, \qquad \frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3 \qquad (123),\ (ABC),\ (pqr)$$

$$(e_1^2 + e_2^2 + e_3^2 = 1,\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \Gamma^2)$$

under conditions

$$e_2 = e_3 = 0, \qquad 9\,(2B-A)\,(2C-A) = 4BC \tag{2.1}$$

$$^1/_8\,(17+\sqrt{73}) > 2B\,/\,A > 3 > 5C\,/\,A > {}^5/_{16}\,(1+\sqrt{73}) \tag{2.2}$$

have the particular solution which has been found by Chaplygin

$$(2B-A)\,(B-C)\,A^{-1}q^2 = (C-A)^2\,p - 3\,(3A-2C)\,sp^{2/3}$$

$$(2C-A)\,(B-C)\,A^{-1}r^2 = (A-B)\,p^2 + 3\,(3A-2B)\,sp^{2/3}$$

$$(2B-A)\,(2C-A)\,\gamma_1 = A\,(B-A)\,(C-A)\,p^2 + {}^3/_2\,A\,(3A^2-4BC)\,sp^{2/3}$$

$$(2C-A)\,\gamma_2 = q\,[(B-A)\,(C-A)\,p + C\,(3A-2B)\,sp^{-1/3}]$$

$$(2B-A)\,\gamma_3 = r\,[(B-A)\,(C-A)\,p + B\,(3A-2C)\,sp^{-1/3}]$$

where

$$s^3 = \frac{4(2B-A)^2\,(2C-A)^2\,\Gamma^2}{9A^3\,(2B+2C-3A)\,(3A-2B)\,(3A-2C)} < 0$$

Chaplygin has also shown the equations of the meridian of the surface of revolution

$$(2B-A)\,(2C-A)\,(p^2+q^2+r^2) = 4\,(B-A)\,(C-A)\,p^2 + 12s\,(2A-B-C)\,Ap^{2/3}$$

$$(2B-A)\,(2C-A)\,(p\gamma_1+q\gamma_2+r\gamma_3) = 6As\,(B-A)\,(C-A)\,p^{5/3} + $$
$$+\,3s^2\,(3A-2B)\,(3A-2C)\,Ap^{1/3}$$

We shall write the solution of Chaplygin in a form convenient for the subsequent analysis.

Satisfying (2.1) we set

$$B = {}^3/_{16}\,A\,(3c+1)\,/\,c, \qquad C = {}^3/_{16}\,A\,(3+c)$$

and by (2.5) we have

$$^1/_3\,(\sqrt{73}-8) < c < {}^1/_5 \tag{2.3}$$

We shall divide the components of the angular velocity by $(-18s)^{\frac{3}{4}}$, introduce the unit vector $\nu_1 = \gamma_1/\Gamma$ and rewrite the solution of Chaplygin as follows:

$$p = \sigma\sqrt{\sigma}, \qquad q = q_*\sqrt{\sigma\,(\sigma^{*2}-\sigma^2)}, \qquad r = r_*\sqrt{\sigma\,(\sigma_*^2-\sigma^2)}$$

$$q_* = 2c\left(\frac{2\,(7-3c)}{3\,(3+c)\,(1-c^2)}\right)^{1/2}, \qquad r_* = 2\left(\frac{2\,(3-7c)}{3\,(1+3c)\,(1-c^2)}\right)^{1/2} \tag{2.4}$$

$$\nu_1 = \frac{(7-3c)(3-7c)\sqrt{3}}{2(1-c)\sqrt{(5-c)(1-5c)}}\,\sigma\left(\frac{9-34c+9c^2}{(7-3c)(3-7c)}-\sigma^2\right)$$

$$\nu_2 = \frac{(3-7c)}{4(1-c)}\left(\frac{(3+c)(7-3c)^3}{2(5-c)(1-5c)(1-c^2)}(\sigma_*^2-\sigma^2)\right)^{1/2}\left(\frac{3+c}{7-3c}\sigma_*^2-\sigma^2\right) \qquad (2.5)$$

$$\nu_3 = -\frac{7-3c}{4(1-c)}\left(\frac{(1+3c)(3-7c)^3}{2(5-c)(1-5c)(1-c^2)}(\sigma_*^2-\sigma^2)\right)^{1/2}\left(\frac{1+3c}{3-7c}\sigma^{*2}+\sigma^2\right)$$

$$\omega^2 = \omega_*^2\sigma\left\{\frac{8}{3}\frac{3-14c+3c^2}{(7-3c)(3-7c)}-\sigma^2\right\}, \qquad \omega_*^2 = \frac{(3-7c)(7-3c)}{(3+c)(1+3c)}$$

$$\omega_\zeta = \omega^*\sqrt{\sigma}\,(\sigma^2-\sigma_*^2\sigma^{*2}), \qquad \omega^* = \frac{(7-3c)(3-7c)}{2(1-c)\sqrt{3(5-c)(1-5c)}} \qquad (2.6)$$

$$\sigma = \left(\frac{1-5c}{3-7c}\right)^{1/2}, \qquad \sigma^* = \left(\frac{5-c}{7-3c}\right)^{1/2} \qquad (2.7)$$

The constants marked by asterisks are positive.

The variable σ is determined from Equation

$$\frac{d\sigma}{d\tau} = \sqrt{\sigma(\sigma_*^2-\sigma^2)(\sigma^{*2}-\sigma^2)}, \qquad \left(\tau = \frac{1}{3}(-18s)^{3/4}\omega_* t\right) \qquad (2.8)$$

The radial component of the angular velocity is determined from (2.6)

$$\omega_\rho = \sqrt{\omega^2-\omega_\zeta^2} = \omega^*\sqrt{\sigma(\sigma_3^2+\sigma^2)(\sigma_4^2-\sigma^2)} \qquad (2.9)$$

where

$$\sigma_3^2 = \frac{4(1-c)R(c)+(3-22c+3c^2)(1-5c)(5-c)}{(3+c)(1+3c)(7-3c)(3-7c)}$$

$$\sigma_4^2 = \frac{4(1-c)R(c)-(3-22c+3c^2)(1-5c)(5-c)}{(3+c)(1+3c)(7-3c)(3-7c)} \qquad (2.10)$$

$$R(c) = \sqrt{2(1-5c)(5-c)(9-42c+34c^2-42c^3+9c^4)}$$

Substituting (2.4), (2.5), (2.9) into (1.6) we have

$$\frac{d\alpha}{d\sigma} = N\frac{(\sigma_1^2-\sigma^2)(\sigma_2^2-\sigma^2)}{(\sigma_3^2+\sigma^2)(\sigma_4^2-\sigma^2)\sqrt{(\sigma^{*2}-\sigma^2)(\sigma_*^2-\sigma^2)}} \qquad (2.11)$$

Here

$$N = 2\sigma_*\sigma^*\sqrt{3(3+c)(1+3c)}$$

$$\sigma_1^2 = \frac{9-34c+9c^2+2(1-c)\sqrt{9-34c+9c^2}}{(7-3c)(3-7c)} \qquad (2.12)$$

$$\sigma_2^2 = \frac{9-34c+9c^2-2(1-c)\sqrt{9-34c+9c^2}}{(7-3c)(3-7c)}$$

Let us write down the inequalities resulting from (2.3), (2.7), (2.12) and (2.10)

$$\sigma_* < \sigma^* < 1, \qquad 0 < \sigma_2 < \sigma_* < \sigma_1, \qquad \sigma_* < \sigma_4 \qquad (2.13)$$

The quantities (2.4) are real, therefore when the body moves the variable σ is bounded

$$0 \leqslant \sigma \leqslant \sigma_* \qquad (2.14)$$

and, as shown by (2.8), it passes from one boundary to another in a finite interval of time. The second derivative $d^2\sigma/d\tau^2$ at $\sigma = 0$ and $\sigma = \sigma_*$ does not vanish, consequently the variable σ cannot remain constant at $\sigma = 0$ or at $\sigma = \sigma_*$. Thus, without any loss of generality we can assume that initially $\sigma = 0$.

3. The curve (2.4) connected with the body constitutes at the same time the moving hodograph of the angular velocity and also the curve of intersection of the cylindrical surfaces

$$^3/_8 c^{-1} (3 + c)(1 - c^2)\, q^2 = (5 - c)\, p^{4/3} - (7 - 3c)\, p^2$$
$$^3/_8 (1 + 3c)(1 - c^2)\, r^2 = (1 - 5c)\, p^{4/3} - (3 - 7c)\, p^2 \qquad (3.1)$$

The directrices of these cylinders are symmetric with respect to the coordinate axes. An example of their forms is shown in Fig.1. To sirplify the graph in Fig.2 there are shown only two loops of the moving hodograph. The remaining two are symmetric to those shown about the plane of symmetry pOr.

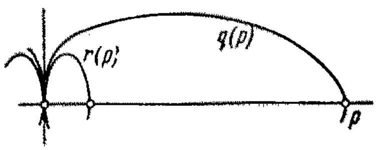


Fig. 1

Let us investigate the motion of the point M which is the tip of the vector ω on the curve (3.1).

From (2.13) and (2.14) follows that the sign of $\sqrt{\sigma^{*2} - \sigma^2}$ does not change and will be assumed positive. Then, from (2.14) and (2.8) follows that near the initial values of σ the radicals $\sqrt{\sigma}$ and $\sqrt{\sigma_*^2 - \sigma^2}$ must have the same sign, which again will be assumed positive. When σ increases from the initial value $\sigma = 0$ to $\sigma = \sigma_*$ the point moves from O through M' to M_*. In M_* the radical $\sqrt{\sigma_*^2 - \sigma^2}$ changes sign, the component r and the derivative $d\sigma / d\tau$ become negative. The variable σ decreases from σ_* to 0, the point M moves from M_* through M'' to O, where the radical $\sqrt{\sigma}$ changes sign, after which p and q become negative and r positive, and the tip of the vector ω in the second half-period moves on the second loop as shown in Fig.2.

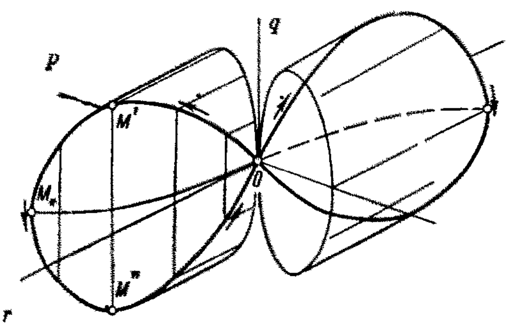


Fig. 2

4. Let us consider the curve (2.6) and (2.9) in the $\omega_\rho\omega_\zeta$ plane. We have

$$\frac{d\omega_\zeta}{d\omega_\rho} = -\frac{(\sigma_*^2\sigma^{*2} - 5\sigma^2)\sqrt{(\sigma_3^2 + \sigma^2)(\sigma_4^2 - \sigma^2)}}{\sigma_3^2\sigma_4^2 - 3(\sigma_3^2 - \sigma_4^2)\sigma^2 - 5\sigma^4} = \tan\vartheta\,(\sigma) \qquad (4.1)$$

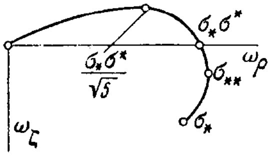
The curve begins from the point O at the angle $\vartheta(0) < 0$, and at $\sigma = \sigma_*\sigma^*/\sqrt{5}$ it has a horizontal tangent at $\sigma = \sigma_*\sigma^* < \sigma_*$ it intersects the $O\omega_\rho$-axis at the angle $\vartheta = \vartheta(\sigma_*\sigma^*) > 0$. Let us note that $\omega_\zeta(\sigma_*) > 0$ and $\vartheta(\sigma_*) < 0$.

At certain value of σ, say $\sigma = \sigma_{**}$, the denominator of the right term of (4.1) vanishes and the tangent to the curve (2.7) and (2.9) is vertical. From the inequalities $\vartheta(\sigma_*\sigma^*) > 0$ and $\vartheta(\sigma_*) < 0$ follows

$$\sigma_*\sigma^* < \sigma_{**} < \sigma_* \qquad (4.2)$$

Fig.3 shows this part if the meridian (2.7) and (2.9) of the surface of revolution which corresponds to the increment of the variable σ from the initial zero value to the value σ^* .

The symmetric image of this line about the point O is the other part



Fif. 3

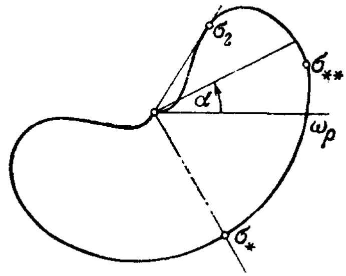


Fig. 4

of the meridian which corresponds to the second half period of the motion of the body.

From (2.9), (2.11) we obtain the projection of the stationary hodograph on the horizontal plane (Fig.4). We have

$$\frac{\omega_\rho d\alpha}{d\omega_\zeta} = 2N \frac{\sigma(\sigma_1^2 - \sigma^2)(\sigma_2^2 - \sigma^2)}{[\sigma_3^2\sigma_4^2 + 3(\sigma_3^2 - \sigma_4^2)\sigma^2 - 5\sigma^4]\sqrt{(\sigma^{*2} - \sigma^2)(\sigma_*^2 - \sigma^2)}} \qquad (4.3)$$

Besides

$$\sigma_*^2\sigma^{*2} - \sigma_2^2 = \frac{2(1-c)(5-c)(1-5c)}{(7-3c)(3-7c)[2(1-c)+\sqrt{9-34c+9c^2}]} > 0 \qquad (4.4)$$

and from (4.4)

$$\sigma_2 < \sigma_{**}$$

Fig. 5 shows one loop of the stationary hodograph, obtained by placing the projection shown on Fig.4 on the surface of revolution with its meridian

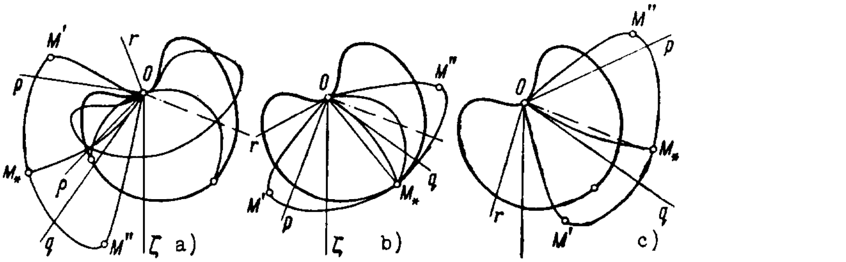


Fig. 5

as shown in Fig.3. Fig.5a shows what happens at the initial value $\sigma = 0$. At this instant, by (2.5) $\nu_1 = 0$, consequently the Op-axis of the body containing the center of gravity is horizontal. When σ increases from 0 to σ_* the moving hodograph passes to the position shown in Fig.5b . In this position $\nu_3 = 0$, and the Or-axis becomes horizontal. When σ decreases from σ_* to 0 the moving hodograph passes to the position shown in Fig.5c. The Op-axis is again horizontal.

In the following motion of the body the point of tangency of the curves shown in Fig.5 moves along these curves in the opposite direction. From the position shown in Fig.5c through the position 5b to the position 5a . The tip of the vector ω describes curves, both in the body and in the space which are symmetric to those shown in Fig.5 about the fixed point O . At the end of this stage of the motion the position 5a is reached again after which the process repeats itself.

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